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1985 J. Phys. A: Math. Gen. 18 1093

(http://iopscience.iop.org/0305-4470/18/7/018)

J. Phys. A: Math. Gen. 17 (1985) 1093–1109. Printed in Great Britain

Critique of the replica trick

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Received 6 August 1984

Abstract. It is shown that the replica trick fails to give the correct non-perturbative result for the two-point function S_2 of the Gaussian unitary ensemble of $N \times N$ random matrices. The failure arises from an incorrect description of the symmetries of the random-matrix system in the limit $N \to \infty$. The correct description, which involves integration over both non-compact and compact degrees of freedom, is obtained by using the method of superfields. Some implications for the localisation transition in disordered electronic systems and the theory of the quantised Hall effect are suggested.

1. Introduction

The theoretical treatment of statistical phenomena in many areas of physics makes use of random Hamiltonians. Observable quantities are evaluated by averaging over an *ensemble* of physical systems in order to avoid the technically difficult task of performing an energy average (or a spatial average, as the case may be) for a fixed realisation of the disorder. However, the mathematical operation of ensemble averaging is still difficult to perform when we are dealing with 'quenched' averages. In statistical mechanics, for example, we wish to average observables such as the free energy $\log Z$ rather than the partition function Z itself. A well known procedure which has been devised for calculating the average of $\log Z$ (Edwards and Anderson 1975) is the so-called 'replica trick',

$$\overline{\log Z} = \lim_{n \to 0} \frac{\overline{Z^n} - 1}{n}. \tag{1.1}$$

Instead of averaging $\log Z$ directly, one studies first the averaged partition function for a system with n replicas (i.e. $\overline{Z^n}$), and then hopes that physical properties of the random system can be extracted by analytically continuing $n \to 0$. For a Gaussian distribution of the disorder, it is easy to calculate the ensemble average of Z^n. The resulting theory can be studied using the standard (perturbative and non-perturbative) techniques of statistical mechanics and field theory and, in this way, a great deal of insight has been gained into the physics of spin glasses, polymers, disordered electronic systems etc.

Despite its highly successful application to the treatment of disordered systems, the replica trick (1.1) suffers from a serious drawback: it is mathematically ill founded. Knowledge of $\overline{Z^n}$ for all positive integer values of n need not be sufficient for extrapolation to the limit $n = 0$. Indeed, soon after its introduction by Edwards and

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Anderson (1975), an example was found where the replica trick gave unphysical results. Sherrington and Kirkpatrick (SK) (1975) observed that an Ising spin glass with infinite-range interactions, when treated using the replica trick, acquires a negative entropy at low temperatures. (We do not discuss recent attempts to resolve these problems by breaking the symmetry in replica space.) They speculated that this unphysical behaviour might arise from the interchange of the thermodynamic limit and the limit $n \to 0$. van Hemmen and Palmer (1979) have argued, however, that for the SK model it *is* permissible to interchange these limits and that the difficulties are to be attributed to the non-uniqueness of the analytic continuation $n \to 0$. In particular, the continuation process cannot be well defined if $\overline{Z^n}$ is non-analytic at $n = 0$. We completely agree with the mathematical content of the paper by van Hemmen and Palmer. However, a purely mathematical argument centred around (non-)analyticity leaves us somewhat unsatisfied because it gives little insight into the *physical* mechanism that causes the replica trick to break down. The present paper aims to draw attention to the fact that problems with the replica trick may occur whenever the theory for a general integer value of n does not have the same *symmetries* as the theory for $n = 0$. (The precise meaning of this statement will become clear in § 2.)

Our personal experience with the replica trick stems from its application to the study of correlation properties of eigenvalues of random matrices. Spectral correlations of random-matrix ensembles are of considerable interest in statistical nuclear theory and the physics of small chaotic systems ('quantum chaos'). From the viewpoint of replica theory, spectral n-point correlation functions represent a fertile testing ground because in some simple cases these functions can be evaluated analytically by quite different methods. We have used the replica trick (Verbaarschot and Zirnbauer 1984) to rederive the perturbation expansion in r^{-1} for the spectral two-point function $S_2(r)$ of the Gaussian orthogonal ensemble. (With d the mean spacing, $S_2(r)$ measures correlations between eigenvalues that are separated by an average of r/d eigenvalues. For definitions see Verbaarschot and Zirnbauer (1984) and § 2 of the present paper.)

More recently we discovered, rather to our surprise, that the replica formalism permits also a non-perturbative evaluation of $S_2(r)$. (In the literature claims have been made (Efetov 1983) that attempts at such an evaluation meet with unsurmountable difficulties.) We will show in the sequel that the outcome of this calculation *disagrees* with the exact result first derived by Dyson (1962a). A mere demonstration of the discrepancy would not deserve particular mention as it only adds to existing knowledge concerning problems with the replica trick. What we do consider of great interest, not only to workers in the field of random-matrix physics but to a much wider audience, is that we can actually perceive the physical reason for the failure. Our analytic evaluation of $S_2(r)$ sheds new light on the intricacies and pitfalls of the replica trick, and this insight we wish to communicate in the present paper.

Our result is summarised as follows. The spectral two-point function $S_2(r)$ for the Gaussian unitary ensemble (GUE) is evaluated as

$$S_2(r) = 1 - r^{-2} \tag{1.2a}$$

using replicated commuting variables, and as

$$S_2(r) = \lim_{n\to 0} n^{-2} \int_{-1}^{+1} \left(\sum_{k=1}^{n} \lambda_k\right)^2 \exp\left(ir \sum_{m=1}^{n} \lambda_m\right) \prod_{m_1<m_2} (\lambda_{m_1} - \lambda_{m_2})^2 \prod_{l=1}^{n} d\lambda_l \tag{1.2b}$$

using replicated anticommuting variables (Grassmann variables.) The correct result is given by

$$S_2(r) = 1 + 2\mathrm{i}r^{-2}\,\mathrm{e}^{\mathrm{i}r}\sin r. \tag{1.2c}$$

All results are derived by reduction to a zero-dimensional nonlinear σ-model. The formulation in terms of replicated commuting variables leads to a nonlinear σ-model with non-compact ('hyperbolic') symmetry (Wegner 1979), while the corresponding model obtained by using Grassmann variables is characterised by a compact ('elliptic') symmetry (Efetov *et al* 1980, Pruisken 1984). Neither of these symmetries gives a correct description of GUE eigenvalue correlations. This is most clearly seen for small values of r where equation (1.2*a*) diverges as r^{-2}, (1.2*c*) as r^{-1} and (1.2*b*) tends to a constant. The correct integral representation of $S_2(r)$ uses both commuting *and* anti-commuting variables ('superfields') and can be reduced to a 'nonlinear supermatrix σ-model' (Efetov 1983). The pseudo-unitary graded Lie group associated with this model involves bosonic as well as fermionic degrees of freedom, of which the former are partly compact and partly non-compact, thus yielding a highly complex and fascinating geometric setting.

The essential steps in the derivation of equations (1.2) are given in § 2, with special emphasis on convergence and symmetry considerations. In § 3 we discuss some implications of equations (1.2) for the localisation transition in disordered electronic systems and the theory of the quantised Hall effect.

2. Evaluation of the GUE two-point function

Dyson (1962b) has argued on quite general grounds that ensembles of random matrices with relevance to physics may have three different types of symmetry, referred to as orthogonal, unitary and symplectic. When the matrix elements are taken to be uncorrelated, Gaussian distributed variables, the resulting ensembles are called the Gaussian orthogonal (GOE), Gaussian unitary (GUE) and Gaussian symplectic ensemble (GSE). The second ensemble applies to systems with broken time-reversal symmetry, and the third and first to time-reversal invariant systems with or without spin-dependent interactions. Since the point we wish to make is of a technical nature, we choose to consider that particular ensemble which has the simplest mathematical properties, namely the GUE.

Information about spectral correlations of random-matrix ensembles is contained in the two-point function S_2,

$$S_2(E_1, E_2) = N^{-2}\,\overline{\mathrm{Tr}(E_1 - H)^{-1}\,\mathrm{Tr}(E_2 - H)^{-1}}. \tag{2.1a}$$

We denote by N the dimension of the Hamiltonian matrix H, and the horizontal bar indicates the average over the ensemble. For large values of N, the connected part of S_2,

$$S_2^{\mathrm{c}}(E_1, E_2) = S_2(E_1, E_2) - N^{-2}\,\overline{\mathrm{Tr}(E_1 - H)^{-1}}\;\overline{\mathrm{Tr}(E_2 - H)^{-1}}, \tag{2.1b}$$

differs from zero only when the (complex) energies E_1 and E_2 lie on opposite sides of the real axis. In the same limit, S_2 becomes a function solely of the 'local' distance variable $r = N(E_1 - E_2)$ measuring $E_1 - E_2$ in units of the local mean spacing $d \propto N^{-1}$.

In the asymptotic regime ($r \to \infty$), S_2 describes long-range correlations between the eigenvalues, related to the 'stiffness' of the spectrum. This is for example seen from an identity for the variance $\Sigma^2(p)$ of the number of eigenvalues in an interval containing p eigenvalues on average (Brody *et al* 1981):

$$\Sigma^2(p) = \mathrm{Re}\int_0^p \mathrm{d}s\,(p-s)S_2^c(\pi s), \qquad s = r/\pi. \tag{2.2}$$

At short range ($r \to 0$), S_2 carries information about the repulsion between neighbouring eigenvalues (level repulsion). For the case of the GOE and the GUE, exact expressions for a related two-point function, $Y_2(r)$, have been given by Dyson (1962a).

Having expounded the physical meaning of S_2, we now proceed to evaluate this function for the GUE by using the replica trick, first with commuting and then with anticommuting variables. (The use of anticommuting variables can be viewed as an extension of the ordinary replica trick to negative integer values of n.) We will see that this calculation provides a simple but non-trivial example where the replica trick fails. At the same time, it illustrates the role of symmetries in bringing about the failure.

2.1. Replicated commuting variables

Introducing $2\times n\times N$ complex replicas $\phi_p^m(c)$ ($p = 1, 2$; $m = 1, 2, \ldots, n$; $c = 1, 2, \ldots, N$), we can represent the GUE two-point function for $\mathrm{Im}\, E_1 > 0 > \mathrm{Im}\, E_2$ as a Gaussian integral (see e.g. Schäfer and Wegner 1980):

$$S_2 = (nN)^{-2}\sum_{abkl}\int \phi_1^{k*}(a)\phi_1^k(a)\phi_2^{l*}(b)\phi_2^l(b)\,\overline{\exp[-\mathscr{L}_1(\phi)]}\,\mathrm{d}[\phi], \tag{2.3a}$$

$$\mathscr{L}_1(\phi) = -\mathrm{i}\sum_{pm}\sum_{cc'}\phi_p^{m*}(c)\sqrt{s_p}(E_p\,\delta_{cc'} - H_{cc'})\sqrt{s_p}\,\phi_p^m(c'), \tag{2.3b}$$

$$s_1 = +1 = -s_2, \qquad \mathrm{d}[\phi] = \prod_{p=1}^{2}\prod_{m=1}^{n}\prod_{c=1}^{N}\mathrm{d}\phi_p^m(c)\,\mathrm{d}\phi_p^{m*}(c). \tag{2.3c}$$

To make contact with equation (1.1) we note that (2.3) can also be written as

$$S_2(E_1, E_2) = N^{-2}\frac{\mathrm{d}^2}{\mathrm{d}E_1\,\mathrm{d}E_2}\overline{\log Z} = (nN)^{-2}\frac{\mathrm{d}^2}{\mathrm{d}E_1\,\mathrm{d}E_2}\overline{Z^n}, \tag{2.4a}$$

$$Z^n = \int\overline{\exp[-\mathscr{L}_1(\phi)]}\,\mathrm{d}[\phi]. \tag{2.4b}$$

According to the replica prescription, we will evaluate the RHS of equation (2.3a) for all positive integers n and then take the limit $n \to 0$. As was stated in § 1, the merit of equation (2.3) derives from the ease with which we perform the average over the Gaussian distribution of H:

$$\overline{\exp[-\mathscr{L}_1(\phi)]} = \mathcal{N}^{-1}\int\exp[-\tfrac{1}{2}N\,\mathrm{Tr}\,H^+H - \mathscr{L}_1(\phi)]\,\mathrm{d}[H] = \exp[-\mathscr{L}_2(\phi)], \tag{2.5a}$$

$$\mathscr{L}_2(\phi) = -\mathrm{i}\sum_p E_p\,\mathrm{Tr}\,S_{pp} + \frac{1}{2N}\sum_{pp'}\mathrm{Tr}\,S_{pp'}S_{p'p}. \tag{2.5b}$$

$$S_{pp'}^{mm'} = \sqrt{s_p}\sum_c\phi_p^m(c)\phi_{p'}^{m'*}(c)\sqrt{s_{p'}}. \tag{2.5c}$$

In order to display clearly the symmetries of (2.5b), we reorganise $\mathscr{L}_2$ and separate it into three parts,

$$\mathscr{L}_2(\phi) = -\mathrm{i}\tfrac{1}{2}(E_1 - E_2)(\phi_1^+\phi_1 + \phi_2^+\phi_2)$$
$$-\mathrm{i}\tfrac{1}{2}(E_1 + E_2)\phi^+ s\phi + \frac{1}{2N}\sum_{ab} \phi^+(a)s\phi(b)\phi^+(b)s\phi(a). \tag{2.6}$$

The first part of $\mathscr{L}_2$ vanishes as $E_1 \to E_2$. In this limit, $\mathscr{L}_2$ acquires an invariance under transformations of the non-compact group $\mathrm{U}(n, n)$ (Wegner 1979) which, adopting Wegner's terminology, we refer to as 'hyperbolic' symmetry. Recognition of hyperbolic symmetry is essential for the rigour of further mathematical treatment. Due to the non-compactness of the parameter space of $\mathrm{U}(n, n)$, the integral in (2.5a) diverges for $E_1 = E_2$, i.e. in the absence of a symmetry-breaking term. The divergence as such is not disconcerting since, as we see from equation (1.2c), $S_2(r)$ does indeed have a genuine singularity at the point $r = N(E_1 - E_2) = 0$. For $E_1 \neq E_2$, the first part of $\mathscr{L}_2$ breaks hyperbolic symmetry and makes the integral convergent.

In addition to hyperbolic symmetry, $\mathscr{L}_2$ has an invariance under unitary transformations in the space of basis vectors $a = 1, 2, \ldots, N$. This invariance suggests the introduction of composite variables σ via the Hubbard–Stratonovich transformation,

$$\exp(-(2N)^{-1}\,\mathrm{Tr}\,S^2) = \exp[-(2N)^{-1}\,\mathrm{Tr}\,(\sqrt{s}\phi\phi^+\sqrt{s})^2]$$
$$= \mathcal{N}^{-1}\int \exp(-\tfrac{1}{2}N\,\mathrm{Tr}\,\sigma^2 - \mathrm{i}\phi^+\sqrt{s}\sigma\sqrt{s}\phi)\,\mathrm{d}[\sigma]. \tag{2.7}$$

As it stands, the decomposition (2.7) is purely formal because we have not yet specified the choice of integration contour for σ. Two requirements are necessary to make further operations mathematically well defined. The first (and obvious) requirement is that (2.7) must lead to a convergent ϕ-integral. Second, we will eventually interchange the integrations over σ and ϕ, and this is allowed only if the σ integration is *uniformly* convergent in ϕ.

The most simple-minded approach would be to take σ Hermitian. Such a choice fails because it results in a *real* cross-term, $\phi_1^*\sigma_{12}\phi_2 + \phi_1\sigma_{12}^*\phi_2^*$, which is bounded neither from below nor from above, leaving us with a badly divergent ϕ-integral. One might attempt to amend this by making the replacement $\sigma_{12} \to \mathrm{i}\sigma_{12}$, but then the integral over σ_{12} becomes divergent.

Going back to equation (2.6) we realise that all attempts to take σ Hermitian (or trivial modifications thereof) are doomed to fail because they disregard hyperbolic symmetry. (This may seem a trivial point but we elaborate on it to emphasise that the non-compact symmetry of the final expression (2.17a) is actually *dictated* by convergence requirements.) One possible parametrisation for σ is found by observing that hyperbolic transformations on ϕ generate a corresponding transformation on σ,

$$\phi \to \phi' = T_1\phi \Rightarrow \sigma \to \sigma' = T_2^{-1}\sigma T_2,\ T_2 = s^{+1/2}T_1 s^{-1/2}. \tag{2.8}$$

Both T_1 and T_2 are elements of $\mathrm{U}(n, n)$,

$$T_1^+ s T_1 = s, \qquad T_2^+ s T_2 = s. \tag{2.9}$$

Equation (2.8) suggests the following choice for σ:

$$\sigma = TPT^{-1} \tag{2.10}$$

where P is diagonal and T runs through the parameter space of $\mathrm{U}(n, n)$. However, for later analysis it is actually more convenient to take T from the coset space $\mathrm{U}(n,n)/\mathrm{U}(n)\times\mathrm{U}(n)$, and include the remaining degrees of freedom in P (Pruisken and Schäfer 1982):

$$P^+=P, \qquad P^{kk'}_{pp'}=\delta_{pp'}P^{kk'}_p, \tag{2.11a}$$

$$T^+=T, \qquad T\in\mathrm{U}(n,n)/\mathrm{U}(n)\times\mathrm{U}(n). \tag{2.11b}$$

The parametrisation (2.11) implies a change in integration volume,

$$\mathrm{d}[\sigma]=I(P)\,\mathrm{d}[P]\,\mathrm{d}\mu(T), \tag{2.12a}$$

$$I(P)=\prod_{k,k'}(\Lambda_{1,k}-\Lambda_{2,k'})^2, \tag{2.12b}$$

$$\mathrm{d}[P]=\left(\prod_k \mathrm{d}P_{kk}\right)\left(\prod_{k<k'}\mathrm{d}P_{kk'}\,\mathrm{d}P^*_{kk'}\right) \tag{2.12c}$$

where $\Lambda_{p,k}$ are the eigenvalues of P_p. $\mathrm{d}\mu(T)$ is the invariant measure of the coset space $\mathrm{U}(n,n)/\mathrm{U}(n)\times\mathrm{U}(n)$ and depends on the particular representation of T used. If we choose

$$T=\begin{pmatrix}(1+t^+t)^{1/2} & t^+\\ t & (1+tt^+)^{1/2}\end{pmatrix}, \tag{2.13}$$

then $\mathrm{d}\mu(T)$ takes the astonishingly simple form

$$\mathrm{d}\mu(T)=\prod_{k,k'}\mathrm{d}t_{kk'}\,\mathrm{d}t^*_{kk'}. \tag{2.14}$$

The discovery of equation (2.14) first suggested to us that an exact evaluation of S_2 within the replica formalism should be possible.

With the parametrisation (2.11) the σ integral converges uniformly in ϕ provided we shift the integration contours for the diagonal part of P off the real axis, $P^{kk}_p\to P^{kk}_p-\mathrm{i}s_p\nu$ ($\nu>0$), and add an infinitesimal symmetry-breaking term $-\mathrm{i}\eta\,\mathrm{Tr}\,\sigma s$ ($\eta=0+$) in the exponent. The Gaussian integration over ϕ is also convergent and can be executed to give

$$S_2=n^{-2}\int \mathrm{Tr}\,\sigma_{11}\,\mathrm{Tr}\,\sigma_{22}\exp[-\mathscr{L}_3(\sigma)]\,\mathrm{d}[\sigma], \tag{2.15a}$$

$$\begin{aligned}\mathscr{L}_3(\sigma)&=\tfrac{1}{2}N\,\mathrm{Tr}(\sigma+\eta/N)^2+N\,\mathrm{Tr}\log(E-\sigma)\\ &=\tfrac{1}{2}N\,\mathrm{Tr}\,P^2+\mathrm{Tr}\,P(T^{-1}\eta T)+N\,\mathrm{Tr}\log(E-P),\end{aligned} \tag{2.15b}$$

$$E=\tfrac{1}{2}(E_1+E_2), \qquad \eta_p=N(E_p-E). \tag{2.15c}$$

Partial integrations have been used to convert the pre-exponential factor in equation (2.3a) into that given in (2.15a). In (2.15b) we have also introduced the 'local' differences η_p which are held fixed in the limit $N\to\infty$.

The rationale behind using the parametrisation (2.11) is now clear from (2.15b): the integration over T requires non-perturbative treatment (for small η), while P can be approximated by the solution $P^{kk'}_{pp'}=\delta_{pp'}\,\delta_{kk'}P^0_p$ of the saddle-point equation $P^0=(E-P^0)^{-1}$. This approximation becomes exact for $N\to\infty$ if we include the contribution from quadratic fluctuations around the saddle point $P=P^0$.

To keep the remaining steps as transparent as possible we take E in the centre of the spectrum ($E=0$). In this case the solution of the saddle-point equation is given by $P_2^0 = \mathrm{i} = -P_1^0$. (Note that this is the *only* saddle point that can be reached without crossing the hypersurface defined by the singularities of $\log(E-P)$.) The expression for S_2 now reduces to

$$S_2 = \text{constant} \times n^{-2} \int \operatorname{Tr} \sigma_{11} \operatorname{Tr} \sigma_{22}|_{P=P^0} \exp[\mathrm{i} \operatorname{Tr}(TsT^{-1}\eta)] \, \mathrm{d}\mu(T). \quad (2.16)$$

Evaluation of the Gaussian integral over $\delta P = P - P^0$ yields an irrelevant factor which has been included in the overall constant, together with another factor originating from the measure term $I(P)$ in (2.12*b*). We now proceed by transforming to the eigenvalues of $\mathrm{i}\sigma_{11} = (TsT^{-1})_{11}$ as independent variables of integration. Calculating the change in the measure, and integrating over angles, we obtain

$$S_2(r) = \text{constant} \times n^{-2} \int_1^\infty \bar{\lambda}^2 \mathrm{e}^{\mathrm{i}r\bar{\lambda}} \prod_{m_1<m_2} (\lambda_{m_1} - \lambda_{m_2})^2 \prod_{m=1}^{n} \mathrm{d}\lambda_m, \quad (2.17a)$$

$$\bar{\lambda} = \sum_{m=1}^{n} \lambda_m, \qquad r = \eta_1 - \eta_2. \quad (2.17b)$$

This expression for S_2 still contains both connected and disconnected parts, the latter of which we eliminate by displacing the pre-exponential factor as $\bar{\lambda} \to \bar{\lambda} - n$. Finally, we shift the lower integration bound to zero. The resulting integral can be evaluated exactly using the orthogonal polynomial method of Mehta and Gaudin (1960). However, it is unnecessary to go through this calculation because a simple rescaling of variables $(\lambda \to \lambda/r)$ shows that

$$S_2^c(r) = -c(n) r^{-2-n^2} \xrightarrow{n \to 0} -r^{-2}. \quad (2.18)$$

The constant $c(n)$ goes to unity for $n \to 0$, as follows from the requirement $\lim_{n\to 0} \overline{Z^n} = 1$. ($\overline{Z^n}$ is obtained from (2.17*a*) by omitting the pre-exponential factor $\bar{\lambda}^2$ and the factor n^{-2}.) Equation (2.18) constitutes the final result of this subsection.

We have known for some time that r^{-2} is the first term of the (asymptotic) r^{-1} expansion for the GUE two-point function, as generated by the replica method. We also knew that low-order corrections vanish identically (this stands in contrast with the GOE two-point function, which has non-zero corrections), but now we see that the cancellation holds even beyond the level of perturbation theory. This is disastrous because equation (2.18) is definitely incorrect. We will return to this point in § 3.

2.2. *Replicated anticommuting variables*

On a purely formal level, the disagreement between equations (1.2*a*) and (1.2*c*) can be blamed on the analytic continuation $n \to 0$, which need not be unique. This explanation leaves unanswered the question as to *why* the replica trick gives the wrong result for S_2 but perfectly reasonable and correct results for the one-point function (Verbaarschot and Zirnbauer 1984). To get some insight into this problem we use the same method of analysis as in § 2.1, but represent now the generating function $\log Z$ as the zero-component limit of a Grassmann integral. As was mentioned earlier, this corresponds to extending $\overline{Z^n}$ to negative integer values of n. The formal treatment using anticommuting variables is quite similar to that using commuting variables, and we give only a brief account of the most important modifications.

We set out from

$$S_2=(nN)^{-2}\sum_{abkl}\int \chi_1^{k*}(a)\chi_1^k(a)\chi_2^{l*}(b)\chi_2^l(b)\,\overline{\exp[-\mathscr{L}_1(\chi)]}\,\mathrm{d}[\chi], \tag{2.19a}$$

$$\mathscr{L}_1(\chi)=\sum_{pm}\sum_{cc'}\chi_p^{m*}(c)(E_p\,\delta_{cc'}-H_{cc'})\chi_p^m(c'), \tag{2.19b}$$

$$d[\chi]=\prod_{p=1}^{2}\prod_{m=1}^{n}\prod_{c=1}^{N}\mathrm{d}\chi_p^m(c)\,\mathrm{d}\chi_p^{m*}(c), \tag{2.19c}$$

where χ and χ^* are 'anticommuting c-numbers' (Grassmann variables). In contrast with equation (2.3b), no factors of i in the exponent are needed for convergence. Due to the absence of such factors, the ensemble-averaged integrand for $E_1=E_2$ now has a compact U($2n$) symmetry ('elliptic' symmetry), instead of the earlier U(n, n) symmetry. This essential difference will be seen to persist throughout the calculation and survive in the limit $n\to 0$.

An important simplification related to compact symmetry is that the Hubbard-Stratonovich transformation (2.7) can now be carried out with the simple (and natural) Hermitian choice for σ. There is, however, a subtle point here which needs further discussion. In § 2.1 we were actually *compelled* by convergence requirements to choose a parametrisation such as (2.11), leading to a non-compact manifold of saddle points. In the present case the Hermitian choice for σ is made *for convenience* and not forced by arguments of convergence. In fact, we could use the parametrisation of § 2.1 instead. This apparent ambiguity will be resolved below equation (2.23).

Performing the same formal manipulations as in § 2.1 we easily arrive at

$$S_2=n^{-2}\int \mathrm{Tr}\,\sigma_{11}\,\mathrm{Tr}\,\sigma_{22}\exp[-\tfrac{1}{2}N\,\mathrm{Tr}\,\sigma^2+N\,\mathrm{Tr}\log(E-\mathrm{i}\sigma)]\,\mathrm{d}[\sigma]. \tag{2.20}$$

The integration now extends over the set of all Hermitian matrices σ.

The next step, which we shall give explicitly, is to discuss the continuous manifold of saddle points of the integrand in equation (2.20) for $E_1=E_2=0$. (For simplicity, we again specialise to the centre of the spectrum.) Expressed in terms of the eigenvalues $\lambda_{p,m}$ of σ, the saddle-point equation for (2.20) reads

$$\lambda_{p,m}=1/\lambda_{p,m}. \tag{2.21}$$

In the limit of interest ($N\to\infty$), the Jacobian generated by the diagonalisation of σ does not perturb the position of the saddle points and has been ignored. Equation (2.21) has two real solutions $\lambda=\pm1$. The saddle-point manifold of (2.20) is therefore given by the set of Hermitian matrices σ with eigenvalues ±1. Clearly, this manifold is compact and consists of several disconnected pieces, characterised by the number of eigenvalues $\lambda=+1$, say. Due to the absence of singularities in the integrand, all submanifolds are accessible, and we have no guidance other than physical intuition as to which to choose. In what follows we will only consider the submanifold with n eigenvalues $\lambda=+1$ and $\lambda=-1$ each, which has the maximum dimensionality. (Inclusion of other pieces does not affect the essential point we are trying to make, namely that the integral is finite for all values of the symmetry-breaking term $r=N(E_1-E_2.)$. This manifold can be parametrised as ($T\in \mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$)

$$\sigma=TsT^{-1}\equiv\begin{pmatrix}u_1 & \emptyset\\ \emptyset & u_2\end{pmatrix}\Lambda\begin{pmatrix}u_1^+ & \emptyset\\ \emptyset & u_2^+\end{pmatrix} \tag{2.22}$$

where

$$\boldsymbol{\Lambda} = \begin{pmatrix} \cos\boldsymbol{\theta} & \sin\boldsymbol{\theta}\, e^{i\boldsymbol{\phi}} \\ \sin\boldsymbol{\theta}\, e^{-i\boldsymbol{\phi}} & -\cos\boldsymbol{\theta} \end{pmatrix} \tag{2.23a}$$

is diagonal in each block (pp'),

$$\boldsymbol{\theta} = \operatorname{diag} \theta_k, \qquad \boldsymbol{\phi} = \operatorname{diag} \phi_k \qquad (k = 1, 2, \ldots, n), \tag{2.23b}$$

and u_1 and u_2 are unitary $n \times n$ matrices.

We are now in a position to motivate better the Hermitian choice for σ made above equation (2.20). We recall that the parametrisation (2.11) gives a convergent σ-integral only if we shift P_p^{kk} off the real axis and add an infinitesimal symmetry-breaking term. This operation is consistent with the saddle-point condition for the case of commuting replicas but *inconsistent* with equation (2.21). More precisely, for $E_1 = E_2 = 0$ the infinitesimal symmetry-breaking term $\mathrm{i}s\eta$ induces phase oscillations in the integrand which become ever more rapid as we leave the point $T(t=0)=1$. In other words, the integration in contours for T obtained by restricting equations (2.10) and (2.11) to the saddle-point manifold of equation (2.20) (i.e. by putting $P_p^{kk'} = \delta_{kk'} P_p^0$; $P_1^0 = \pm 1 = -P_2^0$), *do not follow the direction of steepest descent* with respect to the symmetry-breaking term. This analysis justifies the Hermitian choice for σ and shows that compact $\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$ symmetry is the true and *unique* symmetry of the present problem. (Incidentally, it also implies that Wegner's algebraic derivation of symmetry relations (Wegner 1983) for the Gaussian ensembles is only valid in perturbation theory. The author is, of course, aware of the restricted validity of his proof.)

Subsequent analysis is closely analogous to that given in § 2.1. Omitting all the details we go straight to the final result,

$$S_2(r) = \lim_{n\to 0} n^{-2} \int_{-1}^{+1} \bar{\lambda}^2 e^{ir\bar{\lambda}} \prod_{m_1<m_2} (\lambda_{m_1} - \lambda_{m_2})^2 \prod_{m=1}^{n} d\lambda_m, \tag{2.24}$$

which differs from equation (2.17a) by the location of the bounds of integration. This is the only, but essential, difference.

Expression (2.24) is much harder to evaluate than (2.17a) because the orthogonal polynomials associated with the weight $e^{-r\lambda}$ $(r \to ir)$ over the interval $[-1, +1]$ do not belong to the class of 'classical' orthogonal polynomials. (A simplification occurs for $r = 0$, where these orthogonal polynomials reduce to Legendre polynomials.) Nevertheless, equation (2.24) allows us to make several important statements. (i) For $r \to \infty$, the compactness of the integration interval becomes essentially unimportant and thus equation (2.24) gives the correct asymptotic behaviour $\sim r^{-2}$ in this limit. (ii) The method of Mehta and Gaudin (1960) can be used to evaluate expression (2.24) at the point $r = 0$. We find the result $S_2(0) = Z^n (4n^2 - 1)^{-1}$ which, although positive for all integers $n = 1, 2, \ldots$, becomes negative at $n = 0$. This cannot be correct because the general analytic properties of S_2 require that $\operatorname{Re} S_2(0) > 0$. (In fact, $\operatorname{Re} S_2(r) \to +\infty$ for $r \to 0$.) The finiteness of $S_2(0)$ is due to the finiteness of the integration area, which in turn results, for $N \to \infty$, from the compact $\mathrm{U}(2n)$ symmetry that was inherent to the formulation right from the very outset. (iii) Comparison with § 2.1 shows that the systems obtained by choosing a positive or negative number of replicas differ drastically, and irreconcilably, by their symmetry properties. This indicates the impossibility of extrapolating to the limit $n = 0$.

We conclude this subsection by mentioning that the present formulation leads to difficulties already in the calculation of the one-point function, due to an ambiguity in the choice of saddle point. In this sense, we might say that the 'performance' of anticommuting replicas is even worse than that of commuting ones.

2.3. *Method of superfields*

In this section we show how to evaluate the GUE two-point function correctly, by avoiding the replica trick and using the method of superfields instead. The following calculation is to a large extent a synthesis of mathematical steps given in §§ 2.1, 2.2 and so we can afford to be brief without much loss of clarity. We acknowledge that our analysis was inspired by an ingenious but mystifying article of Efetov (1983). A very detailed and complete exposition of the method will be given in another publication (Verbaarschot *et al* 1985).

Introducing a graded vector (or 'supervector') Φ composed of one (ordinary) complex and one Grassmann variable,

$$\Phi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \qquad \Phi^+ = (\phi^* - \chi^*), \tag{2.25}$$

we can express S_2 as

$$S_2 = N^{-2} \sum_{ab} \int \phi_1^*(a)\phi_1(a)\phi_2^*(b)\phi_2(b)\, \overline{\exp[-\mathscr{L}_1(\Phi)]}\, \mathrm{d}[\Phi], \tag{2.26a}$$

$$\mathscr{L}_1(\Phi) = -\mathrm{i} \sum_p \sum_{cc'} \Phi_p^+(c)\sqrt{s_p}(E_p\,\delta_{cc'} - H_{cc'})\sqrt{s_p}\Phi_p(c'). \tag{2.26b}$$

Our choice of normalisation for the Grassmann integral is $\int \chi\, \mathrm{d}\chi = 1/(2\pi)^{1/2}$. We find it convenient to use the adjoint of the second kind (Rittenberg and Scheunert 1978, Efetov 1983), together with the corresponding convention for matrix transposition.

If we define a 'dyadic' product $S_{pp'}$ by

$$S_{pp'} = \sqrt{s_p} \sum_c \Phi_p(c)\Phi_{p'}^+(c)\sqrt{s_{p'}}, \tag{2.27}$$

then the ensemble-averaged exponential $\overline{\exp(-\mathscr{L}_1)}$ can be written in the form $\exp(-\mathscr{L}_2)$ where

$$\mathscr{L}_2(\Phi) = -\mathrm{i} \sum_p E_p \,\mathrm{Trg}\, S_{pp} + \frac{1}{2N} \sum_{pp'} \mathrm{Trg}\, S_{pp'}S_{p'p}, \tag{2.28}$$

and Trg denotes the graded trace (supertrace). The pseudounitary graded symmetry of $\mathscr{L}_2$ for $E_1 = E_2$ has been discussed by Wegner (1983). The Hubbard-Stratonovich transformation (2.7) now requires the introduction of *graded* composite variables σ, which we parametrise as

$$\sigma = \begin{pmatrix} \hat\sigma_{11} & \hat\sigma_{12} \\ \hat\sigma_{21} & \hat\sigma_{22} \end{pmatrix}, \tag{2.29a}$$

$$\hat\sigma_{11} = \begin{pmatrix} \sigma_{11} & \eta_{11} \\ \eta_{11}^* & \mathrm{i}\tau_{11} \end{pmatrix}, \qquad \hat\sigma_{12} = \begin{pmatrix} a & \mathrm{i}\eta_1 \\ \eta_2^* & \mathrm{i}b^* \end{pmatrix},$$

$$\hat\sigma_{21} = \begin{pmatrix} a^* & \eta_2 \\ \mathrm{i}\eta_1^* & \mathrm{i}b \end{pmatrix}, \qquad \hat\sigma_{22} = \begin{pmatrix} \sigma_{22} & \eta_{22} \\ \eta_{22}^* & \mathrm{i}\tau_{22} \end{pmatrix}. \tag{2.29b}$$

Here, all entries on the diagonal of $\hat{\sigma}_{pp'}$ are commuting and entries on the off-diagonal positions are anticommuting variables. All variables are independent. τ_{11} and τ_{22}, which we take as real, are multiplied by factors of i to cancel the minus sign from the graded trace and obtain convergent integrals. We note that the variables τ_{11}, τ_{22}, b and b^* on the fermionic–fermionic block form i times a Hermitian matrix (see § 2.2). Again for reasons of convergence, the integration contour in σ_{11} and σ_{22} cannot be conducted along the real axis (hyperbolic symmetry!) and is modified in accordance with the discussion in § 2.1. For the parametrisation (2.29*b*) $\hat{\sigma}_{12}$ and $\hat{\sigma}_{21}$ are related through

$$\hat{\sigma}_{12}^{+}=s\hat{\sigma}_{21},\qquad \hat{\sigma}_{21}^{+}=\hat{\sigma}_{12}s,\qquad s=\begin{pmatrix}+1 & 0\\ 0 & -1\end{pmatrix}. \tag{2.30}$$

This was the motivation for inserting the factor of i in front of η_1 in equation (2.29*b*).

Executing the Gaussian integrations over Φ, we again obtain an expression like equation (2.15), with the obvious replacements $n^{-1}\operatorname{Tr}\sigma_{11}\rightarrow\sigma_{11}$, $n^{-1}\operatorname{Tr}\sigma_{22}\rightarrow\sigma_{22}$, Tr$\rightarrow$ Trg. To discuss the saddle-point manifold of the resulting Lagrangian for $E_1=E_2=0$, we first replace the Grassmann variables η_1, η_1^*, η_2 and η_2^* by zero. The discussion then reduces to the previous discussion given in §§ 2.1, 2.2. Putting back the Grassmann degrees of freedom, we see that the saddle-point condition is satisfied by $\sigma=-\mathrm{i}TsT^{-1}$ where T is obtained from (2.13) by making the substitutions

$$t^{+}\rightarrow\hat{\sigma}_{12},\qquad t\rightarrow\hat{\sigma}_{21}. \tag{2.31}$$

An equivalent parametrisation of the saddle-point manifold (Efetov 1983) is given by

$$\sigma=\begin{pmatrix}u & \emptyset\\ \emptyset & v\end{pmatrix}\begin{pmatrix}-\mathrm{i}\lambda_1 & 0 & \mu_1 & 0\\ 0 & -\mathrm{i}\lambda_2 & 0 & \mathrm{i}\mu_2^*\\ \mu_1^* & 0 & +\mathrm{i}\lambda_1 & 0\\ 0 & \mathrm{i}\mu_2 & 0 & +\mathrm{i}\lambda_2\end{pmatrix}\begin{pmatrix}u^{-1} & \emptyset\\ \emptyset & v^{-1}\end{pmatrix}, \tag{2.32a}$$

$$\lambda_1=\cosh\theta_1,\qquad \lambda_2=\cos\theta_2,\qquad \mu_1=\sinh\theta_1\,\mathrm{e}^{\mathrm{i}\phi_1},\qquad \mu_2=\sin\theta_2\,\mathrm{e}^{\mathrm{i}\phi_2}, \tag{2.32b}$$

$$u=1+\boldsymbol{\alpha}+\tfrac{1}{2}\boldsymbol{\alpha}^2,\qquad v=1+\mathrm{i}\boldsymbol{\beta}-\tfrac{1}{2}\boldsymbol{\beta}^2,$$

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \alpha^*\\ \alpha & 0\end{pmatrix},\qquad \boldsymbol{\beta}=\begin{pmatrix}0 & \beta^*\\ \beta & 0\end{pmatrix}. \tag{2.32c}$$

The ranges of integration for the ordinary parts of the commuting variables are $0<\theta_1<\infty, 0<\theta_2\leqslant\pi, 0<\phi_1, \phi_2\leqslant 2\pi$. Equation (2.32) uncovers the true and fascinating symmetry of the random-matrix system for $N\rightarrow\infty$: it contains both compact and non-compact degrees of freedom, and these are joined together by odd elements of a Grassmann algebra!

A remark concerning the precise definition of λ_i and μ_i $(i=1,2)$ is in order here. Since these variables arise from diagonalisation of a graded matrix, they contain nilpotent elements of the Grassmann algebra, in addition to the ordinary part. If we choose λ_1 and λ_2 as the new variables of integration (as we do below), then these nilpotent parts must be eliminated by a 'deformation' of the integration contour. This non-trivial point leads to complications (see below) left totally unmentioned by Efetov (1983).

As before, the integration over fluctuations around the saddle-point manifold yields a trivial factor (in this case just unity) if we transform to the representation given in equation (2.11), with the appropriate modifications in the definition of P and T taking into account the symmetries of σ. It also remains true that the invariant measure of the coset space formed by the matrices T equals unity. Transforming back to the elements of $\hat{\sigma}_{12}$ (and $\hat{\sigma}_{21}$) as independent variables of integration, we obtain

$$S_2(r) = \int \sigma_{11}\sigma_{22} \exp[\mathrm{i}r(\lambda_1 - \lambda_2)]\, \mathrm{d}\mu(\sigma), \tag{2.33}$$

where the invariant measure now takes the form

$$\mathrm{d}\mu(\sigma) = \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1\lambda_2} \mathrm{d}a\, \mathrm{d}a^*\, \mathrm{d}b\, \mathrm{d}b^*\, \mathrm{d}\eta_1\, \mathrm{d}\eta_1^*\, \mathrm{d}\eta_2\, \mathrm{d}\eta_2^*. \tag{2.34}$$

Finally, we use Efetov's clever trick (Efetov 1983) of choosing the 'eigenvalues' λ_1 and λ_2 as integration variables. (Without this trick, the evaluation of equation (2.33) is still a daunting job and probably hopeless without the aid of algebraic computer programs.) The change in integration volume is given by

$$\mathrm{d}a\, \mathrm{d}a^*\, \mathrm{d}b\, \mathrm{d}b^* = \mathrm{d}\mu_1\, \mathrm{d}\mu_1^*\, \mathrm{d}\mu_2\, \mathrm{d}\mu_2^* = \lambda_1\, \mathrm{d}\lambda_1\, \lambda_2\, \mathrm{d}\lambda_2\, \mathrm{d}\phi_1\, \mathrm{d}\phi_2, \tag{2.35a}$$

$$\mathrm{d}\eta_1\, \mathrm{d}\eta_1^*\, \mathrm{d}\eta_2\, \mathrm{d}\eta_2^* = (\lambda_1^2 - \lambda_2^2)^{-2}\, \mathrm{d}\alpha\, \mathrm{d}\alpha^*\, \mathrm{d}\beta\, \mathrm{d}\beta^*. \tag{2.35b}$$

Equation (2.35*b*) provides a nice example that the transformation properties of Grassmann variables are 'contragredient' to those of ordinary variables. (For ordinary variables the factor in equation (2.35*b*) would have been $(\lambda_1^2 - \lambda_2^2)^{+2} = (|\mu_1|^2 - |\mu_2|^2)^{+2}$.) It remains to express the pre-exponential factor in (2.33) in terms of the new variables of integration:

$$\sigma_{11}\sigma_{22} = [\lambda_1 + \alpha^*\alpha(\lambda_1 - \lambda_2)][\lambda_2 - \beta^*\beta(\lambda_1 - \lambda_2)] \tag{2.36a}$$

$$= -\tfrac{1}{2}(\sigma_{11}^2 + \sigma_{22}^2) - \alpha^*\alpha\beta^*\beta(\lambda_1 - \lambda_2)^2. \tag{2.36b}$$

The decomposition made in the second line yields the decomposition of S_2 into a connected and a disconnected part. To see that, we observe that the integral obtained by replacing $\sigma_{11}\sigma_{22}$ in (2.33) with $-(\sigma_{11}^2 + \sigma_{22}^2)/2$ has the value

$$-\tfrac{1}{2}\langle\sigma_{11}^2 + \sigma_{22}^2\rangle = -\tfrac{1}{2}[(-\mathrm{i})^2 + (+\mathrm{i})^2] = +1 = (-\mathrm{i})(+\mathrm{i}), \tag{2.37}$$

which equals the product of the GUE *one-point* function with its complex conjugate in the centre of the spectrum ($E = 0$). This term is disconnected and exhausts the disconnected contributions to S_2. We can therefore identify the second term in (2.36*b*) as the one that yields the connected part of S_2. It is now highly fortunate (though not accidental) that this term contains the maximum number of Grassmann variables, for this allows us to drop the nilpotent parts of λ_1, λ_2, ϕ_1 and ϕ_2, yielding four simple real integrations. We understand that such a 'deformation' of integration contour could also be made for the terms in $\sigma_{11}\sigma_{22}$ of lower than maximum order in the Grassmann variables. However, for these terms one is forced to undertake a careful analysis of contributions from the singularity in the integrand arising from equation (2.36*b*). The existence of the decomposition (2.35*b*) relieves us of this burden.

The last, and trivial, step is to perform the remaining integrations, yielding

$$S_2(r) = 1 + \int_{+1}^{\infty} \mathrm{d}\lambda_1 \int_{-1}^{+1} \mathrm{d}\lambda_2 \exp[\mathrm{i}r(\lambda_1 - \lambda_2)] = 1 + 2\mathrm{i}r^{-2}\, \mathrm{e}^{\mathrm{i}r} \sin r. \tag{2.38}$$

(Note that the factor of $4\pi^2$ from the integrals over ϕ_1 and ϕ_2 is cancelled by the

Grassmann integrations.) Equation (2.38) represents the correct result for S_2 in the limit $N \to \infty$.

The integral representation (2.38) shows very nicely how the two-point function makes the (expected) crossover from an asymptotic behaviour $\sim r^{-2}$ to the short-range behaviour $\sim r^{-1}$. (One can argue on very general grounds that the real part of the two-point function for *any* random-matrix ensemble must have a δ-function at the origin, Re $S_2(r) \sim \delta(r)$. This requires a simple pole in S_2 at $r=0$.) In the asymptotic regime $r \to \infty$, which in the terminology to be used in § 3 might be called the 'weak-coupling' limit, the compactness of one of the integrations is irrelevant for the qualitative behaviour of S_2. However, for small r ('strong-coupling' limit), compact symmetry is absolutely essential for obtaining the correct analytic properties of S_2.

3. Discussion

The purpose of this paper was to demonstrate explicitly, and conclusively, that for $N \to \infty$ the two-point function $S_2(r)$ of the Gaussian unitary ensemble (GUE) is evaluated incorrectly by combining the replica trick with naive extrapolation to the limit $n=0$. An equivalent statement applies to the m-point functions ($m>1$) for the Gaussian orthogonal and Gaussian symplectic ensembles.

As is seen from equation (2.38), the correct description of GUE eigenvalue correlations involves the combination of one compact and one non-compact integration. The replica trick, however, can only accommodate *either* non-compact symmetry (when commuting variables are used) *or* compact symmetry (when anticommuting variables are used). For this reason, the replica trick gives meaningful results only in the weak-coupling limit $r \to \infty$, where the distinction between compact and non-compact symmetry loses relevance.

It is worth emphasising once more that the problems regarding symmetry are intimately connected, via the saddle-point equation for the composite variables σ, with the limit $N \to \infty$. There is of course little hope that the multi-dimensional integrals appearing in the replica formalism for S_2 can ever be evaluated non-perturbatively for a finite value of N. However, if the result were available, we would expect analytic continuation to $n=0$ to be possible and yield the correct analytic behaviour of S_2 also for $r=0$.

Finally, we wish to discuss some implications of equations (1.2) for the localisation transition in disordered electronic systems. Being nuclear physicists, we do not feel authorised to make definite statements in this field, and so the following discussion is merely intended as a suggestion.

The connection with localisation theory derives from the work of Schäfer and Wegner (1980) and of McKane and Stone (1981). (In fact, the first part of the calculation in § 2.1 is nothing but a specialisation to the zero-dimensional case of the work of Schäfer and Wegner. Note also that since we are dealing with the case of a unitary ensemble, the following remarks apply to the localisation transition in time-reversal non-invariant systems.) These authors have used the replica trick with commuting fields to show that the mobility-edge behaviour of disordered electronic systems in d dimensions is described by a nonlinear σ-model,

$$Z = \int \exp(-g^{-2} S[\sigma])\, \mathrm{d}\mu(\sigma), \tag{3.1a}$$

$$S[\sigma]=\int \operatorname{Tr}(\partial_\mu\sigma)^2\,\mathrm{d}^d x+\text{symmetry-breaking term} \tag{3.1b}$$

$$\sigma\in \mathrm{U}(n,n)/\mathrm{U}(n)\times\mathrm{U}(n), \qquad n=0. \tag{3.1c}$$

Critical exponents for the localisation transition in $2+\varepsilon$ dimensions can be calculated by analysing the weak-coupling renormalisation group for (3.1). This analysis has been carried as far as four-loop order by Hikami (1983). The most stunning outcome of his work is that corrections to the critical exponent for the conductivity of order ε, ε^2 and ε^3 (and probably also of higher order) vanish identically. It is interesting to speculate on the extent to which this result may change when the replica trick is abandoned in favour of the superfield formalism, yielding the (correct) symmetry for σ given in equation (2.31). The only work in this direction which we are aware of is that of Efetov (1983). His results agree with those of Hikami to the order calculated. This is hardly surprising because Efetov, like Hikami, uses the weak-coupling renormalisation group (ε expansion) which gives identical results for the compact and non-compact models, apart from certain sign changes in the β function. (Non-perturbative effects due to the compactness of the fermionic–fermionic sector are not 'seen' by the ε expansion. Of course, the Grassmann fields appearing in the superfield formalism simply take the role of the $n\to 0$ limit in cancelling vacuum graphs.) The extreme weak-coupling limit is realised in exactly two dimensions, with a fixed point at $g=0$, and for this case the models of Efetov and of Schäfer and Wegner should be equivalent in the asymptotic scaling regime.

In three dimensions the effect of compact symmetry seems less clear. To strengthen our case we may invoke the analogy with four-dimensional QCD. Recent Monte Carlo simulations (Seiler *et al* 1984) have shown that compact and 'non-compact' QCD behave quite differently on the lattice. To be sure, both models are asymptotically free, but the non-compact model differs from Wilson's compact theory in that no evidence for confinement has been found for all values of the coupling accessible to numerical simulation. This confirms the (plausible) expectation that compact symmetries are more effective in disordering a system.

Information on the effect of compact symmetry could be gained by inquiring into the role played by topologically non-trivial field configurations. Instantons for the supermatrix σ-model, which exist in two dimensions (see below), are also present in three dimensions, due to the existence of a non-trivial map $S_3\to S_2$ (Wilczek and Zee 1983). We do not know whether these instanton solutions affect the critical behaviour or simply act as a 'background'.

This and related questions could, in principle, be studied via the high-temperature (strong-coupling) series for the two-particle Green function. Unfortunately, such an expansion scheme is difficult to implement due to the non-compact degree of freedom in σ. An alternative approach would be to pass from (3.1) to the Hamiltonian version of the theory, and use the resulting quantum Hamilton operator for developing the strong-coupling series.

More concrete statements can be made about the theory of the quantised Hall effect as recently developed by Levine *et al* (1983) and Pruisken (1984). These authors argue that Green functions for the disordered electronic system can be generated from a nonlinear σ-model even when the system is subjected to a strong magnetic field. The corresponding Lagrangian differs in form from (3.1*b*) only by the addition of a topologically invariant term. It is argued that this topological invariant changes the long-distance behaviour of the system, and, thereby, gives rise to extended states. Such an argument represents an interesting advance because it reconciles the established theory of localisation in two dimensions with the experimental observation of a quantised Hall conductance. However, Levine *et al* use replicated fields for dealing with the disorder and, cautioned by the experience with our zero-dimensional 'toy model', we are reluctant to accept without scrutiny any non-perturbative prediction based on such treatment. Actually, there seems to exist some confusion (Pruisken 1984) as to the proper choice of symmetry for the replica matrix σ. The theoretical derivation has been given for two alternative formulations, differing by the choice of commuting or anticommuting fields in the representation of the generating functional. As is seen from §§ 2.1, 2.2, the second choice leads to a model with $\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$ symmetry, while the first choice gives rise to the corresponding non-compact symmetry, $\mathrm{U}(n,n)/\mathrm{U}(n)\times\mathrm{U}(n)$. Of these symmetries, only the former admits instantons with finite action, which are required for the mechanism leading to the appearance of extended states. (Needless to say, in this situation, Levine *et al* settled for anticommuting replicas leading to the compact model.)

The resolution of this ambiguity is now trivial in the light of what we said earlier. The formalism of Levine *et al* can be put on a sound theoretical basis by replacing replicated Grassmann fields with superfields. Comparison of §§ 2.2 and 2.3 suggests that the derivation of the effective super-Lagrangian requires only minor modifications of Pruisken's argument. From equation (2.32) it is easy to see that the resulting theory allows for topological excitations, which turn out to be equivalent to those of the O(3) nonlinear σ-model. To see this equivalence, we recall that the invariant measure associated with (2.32) is given by

$$\mathrm{d}\mu(\sigma)=\frac{(\lambda_1+\lambda_2)^2}{\lambda_1\lambda_2}\,\mathrm{d}a\,\mathrm{d}a^*\,\mathrm{d}b\,\mathrm{d}b^*\,\mathrm{d}\eta_1\,\mathrm{d}\eta_1^*\,\mathrm{d}\eta_2\,\mathrm{d}\eta_2^*. \tag{3.2}$$

The bosonic–bosonic fields a and a^* are taken from the trivial topological sector ($a, a^*\approx 0$) and can be treated in (weak-coupling) perturbation theory along with the Grassmann fields η_1, η_1^*, η_2 and η_2^*. Retaining the full topology only for the fermionic–fermionic fields b and b^*, the invariant measure (3.2) reduces to

$$\mathrm{d}\mu(\sigma_{\mathrm{FF}})=\lambda_2^{-1}\,\mathrm{d}b\,\mathrm{d}b^*. \tag{3.3}$$

(We note that the numerator of (3.2) serves to make the Grassmann integrations invariant.) The RHS of equation (3.3) is, in fact, the invariant measure on the manifold of matrices

$$\sigma_{\mathrm{FF}}=\begin{pmatrix}\cos\theta & \sin\theta\,\mathrm{e}^{\mathrm{i}\phi}\\ \sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} & -\cos\theta\end{pmatrix}=u\begin{pmatrix}1&0\\0&-1\end{pmatrix}u^+,$$
$$u\in\mathrm{U}(2)/\mathrm{U}(1)\times\mathrm{U}(1), \tag{3.4}$$

and leads to the following topological invariant:

$$I=\int\lambda_2^{-1}\left(\frac{\partial b}{\partial x_1}\frac{\partial b^*}{\partial x_2}-\frac{\partial b^*}{\partial x_1}\frac{\partial b}{\partial x_2}\right)\mathrm{d}^2x. \tag{3.5}$$

With the definitions

$$b=\sin\theta\,\mathrm{e}^{\mathrm{i}\phi}=s_2+\mathrm{i}s_3,$$
$$b^*=\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi}=s_2-\mathrm{i}s_3,\qquad (1-b^*b)^{1/2}=\cos\theta=s_1, \tag{3.6}$$

we obtain the familiar action,

$$S=-g^{-2}\int \operatorname{Tr}(\partial_\mu \sigma_{\mathrm{FF}})^2\,\mathrm{d}^2x=-g^{-2}\int \sum_{a=1}^{3}(\partial_\mu s_a)^2\,\mathrm{d}^2x, \tag{3.7}$$

and topological invariant,

$$I=\int \frac{1}{s_1}\left(\frac{\partial s_2}{\partial x_1}\frac{\partial s_3}{\partial x_2}-\frac{\partial s_2}{\partial x_2}\frac{\partial s_3}{\partial x_1}\right)\mathrm{d}^2x, \tag{3.8}$$

of the O(3) nonlinear σ-model, which has been the subject of intensive study. In particular, instanton solutions for this model have been studied by Belavin and Polyakov (1975) and, more recently, by Gross (1978). The results of these papers will therefore be of direct use in completing the theoretical description of the quantised Hall effect.

A final word of explanation may be helpful. Due to graded symmetry, the existence of instanton solutions has consequences which are more subtle than usual. The action $\int \mathscr{L}\,\mathrm{d}^2x$ is invariant under global graded transformtions,

$$\hat{\sigma}_{12}(x)\to u\hat{\sigma}_{12}(x)v^{-1}, \qquad \hat{\sigma}_{21}(x)\to v\hat{\sigma}_{21}(x)u^{-1}. \tag{3.9}$$

On the other hand, the aforementioned instanton solutions break graded symmetry by their non-trivial topological structure in the fermionic-fermionic sector. This implies that integration over fluctuations around the instanton solution gives identically zero, due to the existence of a massless mode associated with the broken graded symmetry. (Note that the massless mode is a Grassmann mode ('Goldstone fermion') and thus yields zero rather than infinity.) We conclude that instantons do *not* contribute to the partition function and conserve the constraint $Z=1$. The situation is very different when we calculate physical observables such as the two-particle Green function $K(q, E_1, E_2)$. (The definition of K is found in Schäfer and Wegner (1980).) In this case the global graded symmetry of the integrand is broken by the quantity to be averaged, $a^*(q)a(q)$. As a result, the mass of the Goldstone fermion becomes finite and instantons do yield a non-vanishing contribution.

Acknowledgments

We should like to thank Professor H A Weidenmüller for many stimulating discussions and encouraging us to write the present paper. Thanks are also due to Professor F Wegner for a critical reading of the manuscript and helpful discussions.

Note added in proof. Both referees have urged us to include references to Edwards and Jones (1976) and Jones *et al* (1978). These papers first showed that the replica trick gives correct results when applied to the calculation of the one-point function.

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